## Calculus (Tutorial # 2)

## Sequences

- 1. Let  $a_n = \frac{3n+4}{n+5}$ , for  $n \in \mathbb{N}$  then  $\lim_{n \to \infty} a_n = 3$ . Given an  $\epsilon > 0$ , find  $N_{\epsilon} \in \mathbb{N}$  such that  $|a_n 3| < \epsilon$  for all  $n \ge N_{\epsilon}$ .
- 2. (a) Suppose  $a \in \mathbb{Q}$ , then can you find a sequence  $\{a_n\}$  of irrational numbers such that  $\lim_{n \to \infty} a_n = a$ ?
  - (b) Suppose  $a \in \mathbb{R} \setminus \mathbb{Q}$ , then can you find a sequence  $\{a_n\} \subseteq \mathbb{Q}$  such that  $\lim_{n \to \infty} a_n = a$ ?
- 3. Let  $b_n \geq 0$  and  $b_n \to 0$  as  $n \to \infty$ . Let  $\{a_n\} \subseteq \mathbb{R}$  be a sequence such that  $|a_n a| \leq b_n$  for all  $n \geq N \in \mathbb{N}$ . Then using the definition of convergence of sequences, show that  $a_n \to a$  as  $n \to \infty$ .
- 4. Let  $\{a_n\} \subseteq \mathbb{R}$  be such that  $\alpha \leq a_n \leq \beta$  and  $\{a_n\}$  is converges to a then show that  $\alpha \leq a \leq \beta$ .
- 5. Let  $\{a_n\} \subseteq \mathbb{R}$  is such that for each  $\epsilon > 0$ ,  $|a_n a| < 5\epsilon$ , for all  $n \ge N$  where N does not depend on  $\epsilon$ . Then characterize such  $\{a_n\}$ .
- 6. Let  $\{a_n\} \subseteq \mathbb{R}$  be a convergent sequence. Then using the definition of convergence of sequences show that  $\{a_n^k\}$  for some fixed  $k \in \mathbb{N}$  is also convergent. How about  $\{a_n^k\}$ ?
- 7. Find all the possible conditions on  $\{a_n\} \subseteq \mathbb{R}$ , for which  $\{a_n\}$  is convergent if and only if  $\{|a_n|\}$  is convergent.
- 8. Let  $\{a_n\} \subseteq \mathbb{R}$  be such that  $\lim_{k \to \infty} a_{2k} = a$  and  $\lim_{k \to \infty} a_{2k-1} = a$ . Then is it true that  $\{a_n\}$  is convergent?
- 9. Let  $\{a_n\}$  be a sequence such that  $|a_n| \leq \frac{1+n}{1+n+7n^2}$ , for all  $n \in \mathbb{N}$ . Check whether  $\{a_n\}$  is a Cauchy sequence or not.
- 10. Let  $\{a_n\}$  be a sequence such that  $|a_n a_{n-1}| \le c|a_{n-1} a_{n-2}|$ , for some constant 0 < c < 1 and for all  $n \ge 2$ . Show that  $\{a_n\}$  is convergent.
- 11. Let  $\{a_n\}$  be a sequence of positive real numbers. Assume that  $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$ , with l < 1. Prove that  $a_n \to 0$  as  $n \to \infty$ .
- 12. Let  $\{a_n\}$  be a bounded sequence. Assume that  $a_{n+1} \ge a_n 2^{-n}$ . Check the convergence of  $\{a_n\}$ .
- 13. Let  $a_1 = \sqrt{101}$  and define  $a_n = \sqrt{101 + a_{n-1}}$  for  $n \in \mathbb{N}$ . Then check the convergence of  $\{a_n\}$  and also find the limit if it converges.

14. Let  $0 < b_1 < a_1$ . Define the two sequences  $\{a_n\}$  and  $\{b_n\}$  as follow:

$$a_{n+1} := \frac{a_n + b_n}{2}$$
 and  $b_{n+1} := \sqrt{a_n b_n}$ , for  $n \in \mathbb{N}$ .

Then check the convergence of  $\{a_n\}$  and  $\{b_n\}$ . Also find the limit if they converge.

- 15. If  $\{a_n\}$  is a bounded sequence of real numbers and  $\liminf_{n\to\infty} a_n = l$ , prove that there is a subsequence of  $\{a_n\}$  which converge to l. Also, prove that no subsequence of  $\{a_n\}$  can converge to a limit less than l.
- 16. Find the lim sup and lim inf if they exist for the sequences given below. Also check the convergence of them and find the limit if they converge.

(a) 
$$a_n = \sqrt{n^2 + 5n} - n$$

(b) 
$$a_1 = 3$$
 and  $a_{n+1} = 1 - \frac{1}{a_n}$ 

(c) 
$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$$

(d) 
$$\left\{2, -1, \frac{3}{2}, -\frac{1}{2}, \frac{4}{3}, -\frac{1}{3}, \dots\right\}$$

(e) 
$$a_n = \sin\left(\frac{n\pi}{7}\right)$$

(f) 
$$\{1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots\}$$

(g) 
$$a_n = \left(1 + \frac{1}{n}\right)^n$$

(h) 
$$x_1 = 1$$
,  $x_2 = 2$  and  $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$  for  $n \ge 1$ .

(i) 
$$a_n = \frac{\{n\pi\}}{n}$$
 where  $\{n\pi\}$  is the fractional part of  $n\pi$ .

(j) 
$$a_n = \frac{\sin n}{n}$$
 for  $n \in \mathbb{N}$ .